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Ground-state energy calculation in broken SUSYQM by a WKB-like method

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Abstract. Using the exact solutions of the Schrödinger equation previously given by Fröman and Fröman and by Giler *et al*, we calculate the energy of the ground state for polynomial superpotentials with broken supersymmetry and with an arbitrary number of wells. Our method gives a result which, having a general form in accordance with that given by Salomonson and Van Holten and by Giler *et al*, is asymptotically precise, revealing features characteristic of supersymmetric potentials.

1. Introduction

The WKB method provides a very efficient way of calculating some relevant quantities in quantum mechanics. When applied to the problem of finding the energy eigenvalues E it results in an asymptotic expansion in inverse powers of n. Such an expansion, valid in principle for large n, is surprisingly accurate even for lower-lying eigenstates. Its accuracy grows very rapidly with n (Hioe *et al* 1978). In many cases one obtains an asymptotic expansion in integral powers of $\hbar$, valid for any eigenvalue, which is often closely allied to the usual pertrubative expansion (Fröman and Fröman 1965, see also Fedoryuk 1983). This is the case of the single-well potential $V(x)$ with $V''(x_0) > 0$ at the lowest point, x_0, of the well. If the value E can be expanded as a power series in $\hbar$ with a finite radius of convergence, one can reconstruct this series by the WKB method and sum it to produce the exact solution (Bender *et al* 1977). In most cases, however, the series is only asymptotic (Reed and Simon 1979). It follows then that there are contributions which cannot be seen in ordinary WKB/perturbative expansions. As was shown by Balian *et al* (1979) in the context of large-N expansions such contributions can be numerically quite relevant. Moreover the important theoretical problem arises as to the identity of the non-perturbative contributions. Indeed, given any divergent asymptotic expansion one has to apply some summation method (the Borel method, for example) to obtain the finite answer. Any such method should give the same answer for convergent series but for divergent series the results can be, and in general are, method dependent and differ by $O(\hbar^\infty)$ terms. Consequently, to state the problem properly, i.e. to define the non-perturbative, subdominant contributions to the energy $E(\hbar)$ possessing an asymptotic expansion

$$E(\hbar) \sim \sum_{k \geq 0} \alpha_k \hbar^{2k}$$

it is necessary to select the method by which to sum the above series. If $E'(\hbar)$ is the value ascribed by this method to $\Sigma_{k \geq 0} \alpha_k \hbar^{2k}$ one can define the subleading contribution

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as the asymptotic value of $E(\hbar) - E'(\hbar)$ (Balian *et al* 1979). In principle this procedure can be pursued further to give the complete expansion with the successive subdominant contributions (see Balian *et al* (1979) for a very elegant discussion). There are some cases, however, when the non-perturbative contributions can be unambiguously determined, because in such cases the above contributions give rise to effects which, being qualitatively different, can be disentangled from the 'background' of the dominant series contribution. This is, for example, the case for the symmetric double-well oscillator. The perturbative (or equivalently single-well WKB) expansion for each well is exactly the same; all levels are doubly degenerate and no splitting is possible. Any summation method applied to those series would give the same result. However, there are non-perturbative contributions due to tunnelling that produce level splitting of order $\exp(-\text{constant}/\hbar)$. This splitting can be calculated unambiguously in the dominant order as, for example, in Landau and Lifshitz (1965).

As a second example let us consider the metastable states (false vacua). They are stable to any order in the perturbative/single-well WKB expansion; the energy eigenvalues are real in this approximation. Any reasonable summation procedure will preserve this property. Only the non-perturbative corrections contribute to the imaginary part of the energy as they are responsible for the decay. Again they can be calculated in spite of the fact that the dominant contribution to the (real part of the) energy is only known to some crude approximation (in comparison with the imaginary part).

Another example which will be investigated below is provided by supersymmetric quantum mechanics. It is well known that for some cases (SUSY unbroken at the tree level), due to peculiar fermion-boson cancellations, there are non-perturbative corrections to the ground-state energy (GSE). Therefore the perturbative series is trivially convergent—to zero. Consequently one should be able to evaluate the GSE by means of the WKB-like method. We do this below.

Although such calculations seem to be worthwhile on their own it is perhaps more important that they enable us to investigate a general SUSY potential as well as offering a more precise estimate of the GSE than had been obtained previously (Salomonson and Van Holten 1982, Giler *et al* 1985).

We must, however, stress that our approach is more sophisticated than the usual WKB approximation itself. There are at least of two kinds of reason for this. The first is that the usual WKB method is, in principle, the expansion in $\hbar$ when $\hbar \to 0$ together with the constraint $n\hbar = \text{constant}$, i.e. it enables one to calculate rather high energy levels ($n \to \infty$). On the other hand, even if we relax this restriction, i.e. if we put $n = 0$ in the WKB quantisation conditions (see Giler (1988) for a discussion of the validity of this procedure), we simply get zero for the GSE—a result which is still unsatisfactory.

The second reason is that we want to calculate the GSE for the many-well superpotential with an arbitrary number of wells. In such a case it appears to be incorrect to make use of the usual WKB approximation for the wavefunctions to match them when going from one well to another (this point will become clearer in § 4). On the other hand such a matching procedure is necessary in order to establish the proper quantisation rules for the energy. Consequently we are forced to start from the original Fröman and Fröman (1965, hereafter referred to as FF) form of the solution to the Schrödinger equation (SE) from which the WKB formulae follow directly.

However, we should also mention here that the FF form of the solutions to the SE is not the only one used in our calculations of the GSE. We also utilise for this purpose another type of solution to the SE with a supersymmetric potential, as found in our

previous paper (Giler *et al* 1985). These solutions can be defined in the same regions as the FF solutions (see appendix 1, where we quote them briefly). However, in contrast to the latter, which are asymptotic for $\hbar \to 0$, these 'purely' supersymmetric solutions are instead asymptotic for $\hbar \to \infty$. In the case when the energy E vanishes, however, they become relatively simple in their dependence on $\hbar$. Contrary to this the FF solutions still remain rather complicated functions of $\hbar$ even for $E = 0$. Since the point $E = 0$ is where we perform our main calculations, we make extensive use of these simpler, 'purely' supersymmetric solutions.

We consider in this paper a general superpotential $V(x)$ of the polynomial type which generates the supersymmetric Hamiltonian in the usual manner (Witten 1981, Solomonson and Van Holten 1982, Giler *et al* 1985). Applying the FF method to solve the problem of the GSE we arrive at the result

$$E_0 \simeq (\hbar/2\pi)|V''(z_l^+)V''(z_k^-)|^{1/2} \exp[-(2/\hbar)(V(z_l^+) - V(z_k^-))] \tag{1}$$

where z_l^+ and z_k^- denote those maximum and minimum values of $V(x)$ which maximise the difference $V(z_l^+) - V(z_k^-)$ under the condition that the minimum is *always* to the right from the maximum (this kind of asymmetry is directly connected with our choice of the superpotential for which $V(+\infty) = +\infty$). Therefore, although the form of our present answer is the same as previous results (Giler *et al* 1985) its meaning is different. It is seen that the above condition which relates the suitable extrema present in (1) cannot be inferred from the double-well case considered by Salomonson and Van Holten (1982). It also corrects our previous result (Giler *et al* 1985). In fact, it expresses a characteristic property of supersymmetric one-dimensional quantum mechanics. We give a simple explanation for this phenomenon at the end of § 3.

Our paper is organised as follows. In § 2 we describe the FF approach. Then in § 3 the GSE for the many-well SUSY potential is calculated, and our results are discussed in § 4.

2. The Fröman and Fröman form of the solution to the Schrödinger equation

Let us assume for simplicity that the potential $U(x)$ is an entire function of x and real for real x. We also assume that the spectrum is purely discrete, i.e. $U(x) \to +\infty |x| \to +\infty$. The Schrödinger equation

$$[(-\hbar^2/2m)(d^2/dx^2) + U(x)]\psi(x) = E\psi(x)$$

can be written in the form

$$\psi''(x) - q(x)\psi(x) = 0 \tag{2}$$

where $q(x) = 2m(U(x) - E)/\hbar^2$.

Making the substitution

$$\begin{aligned} &\psi_\sigma(x) = q^{-1/4}(x) \exp(\sigma S(x_0, x))\psi_\sigma(x) \\ &S(x_0, x) = \int_{x_0}^{x} q^{1/2}(y)\, dy \qquad \sigma = \pm 1 \qquad q(x_0) = 0 \end{aligned} \tag{3}$$

we obtain an appropriate equation for $\psi_\sigma(x)$. Solving by iterations we get

$$\begin{aligned} \psi_\sigma(x) = 1 + \sum_{n \geq 1} \sigma^n \int_{\gamma^\sigma(x)} dy_1 \int_{\gamma^\sigma(y_1)} dy_2 \ldots \int_{\gamma^\sigma(y_{n-1})} dy_n\, \omega(y_1) \ldots \omega(y_n) \\ \times [1 - \exp(2\sigma S(x, y_1))][1 - \exp(2\sigma S(y_1, y_2))] \ldots [1 - \exp(2\sigma S(y_{n-1}, y_n))] \end{aligned} \tag{4}$$

where

$$\omega(y) = \tfrac{1}{8}[q''(y)/q^{3/2}(y) - \tfrac{5}{4}(q'(y))^2/q^{5/2}(y)]. \tag{5}$$

In (4) the integration path γ^σ (y_i) starts from infinity and goes to y_i in such a way as to ensure that Re $S(x_0, y_{i+1})$ varies monotonically along the whole path. Moreover, Re $S(x_0, y_{i+1}) \to \mp\infty$ for $\sigma = \pm 1$ when $y_{i+1} \to \infty$ along the path.

A very important point is that the series in (4) is convergent (not only asymptotic) provided the paths γ^σ are chosen as above. The asymptotic WKB expansion valid for $\hbar \to 0$ can be obtained from (4). However, equation (4) clearly contains more information than the asymptotic series it produces. We make use of this fact later in this paper.

A convenient way to describe the domains of validity of the representations (3) and (4) is to draw a so-called Stokes graph (SG). Such a graph consists of the Stokes lines (SL) given by Re $S(x_0, x) = 0$ for all roots x_0 of $q(x)$. Every system of SL divides the whole complex x plane into a set of disjoint pieces. Solutions of the form given by (3) and (4) can be constructed in each piece that contains $+\infty$ or $-\infty$ of Re $S(x_0, x)$ and in each such piece it is uniquely (up to a constant) determined by the condition that it must vanish as $x \to \infty$ in this piece. It can be analytically continued, with the help of (3)–(5), to those pieces to which the path can be continued. In all such pieces it grows exponentially. Moreover, any two solutions of the form (3) and (4) coming from different pieces are linearly independent. This fact is the obvious consequence of their asymptotic behaviour described above.

3. The SUSY ground-state energy

We follow here the conventions of Giler *et al* (1985). Let us consider the case of broken (but unbroken at the tree level) SUSYQM given by the superpotential

$$V(x) = \sum_{k=0}^{2n+1} a_k x^k \qquad a_{2n+1} > 0 \tag{6}$$

the zeros of which are real (see figure 1). The corresponding Hamiltonians can be

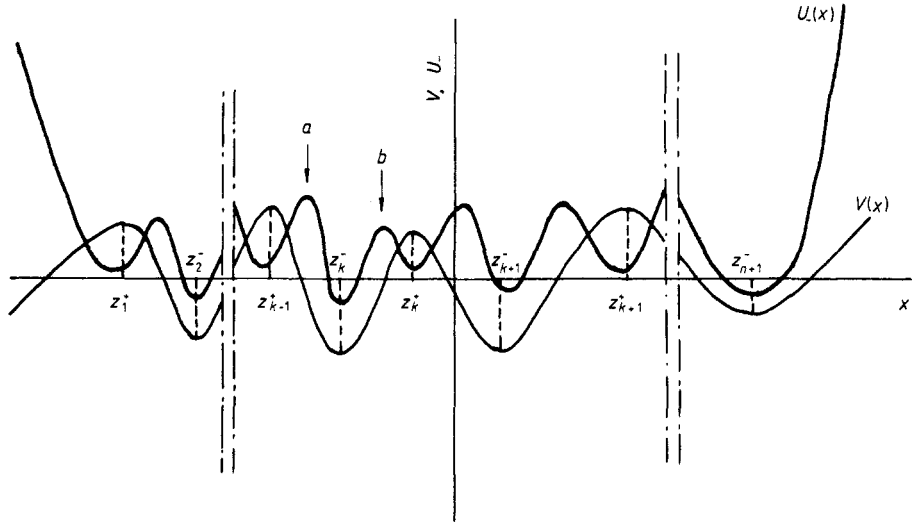

Figure 1. The superpotential $V(x)$ and its potential $U_-(x)$.

written as

$$H_{\mp} = -(\hbar^2/2m)\, \mathrm{d}^2/\mathrm{d}x^2 + \tfrac{1}{2}(V'(x))^2 \mp \tfrac{1}{2}\hbar V''(x). \tag{7}$$

Both these Hamiltonians have the same spectrum. For definiteness we choose H_-. Its potential $U_-(x)$ is depicted in figure 1.

We know (Salomonson and Van Holten 1982, Giler *et al* 1985) that the GSE E_0 is positive and of the order of $\exp(-\text{constant}/\hbar)$. Consequently it lies above the x axis, but below any positive local minimum of $U_-(x)$. Let us denote the real roots of the polynomial $q(x) = 2(U_-(x) - E_0)/\hbar$ by a_k, b_k, with $k = 2, \ldots, n+1$, and denotes its complex conjugate roots by c_k, $\bar{c}_k$, $k = 1, \ldots, n$. Then the SG corresponding to the case under consideration is as sketched in figure 2. It should be noted that the length of the segments (a_k, b_k) and $(c_k, \bar{c}_k)$ are of the order $\hbar^{1/2}$. Further, each zero of $q(x)$ is a branch point for the functions $q^{1/2}(x)$ and $q^{-1/4}(x)$. So the arguments of $q(x)$ written on both sides of the real axis of the x plane serve to give the steps in $q^{1/2}(x)$ and $q^{-1/4}(x)$ at the appropriate intervals of the axis (see figure 4). Let us note also that

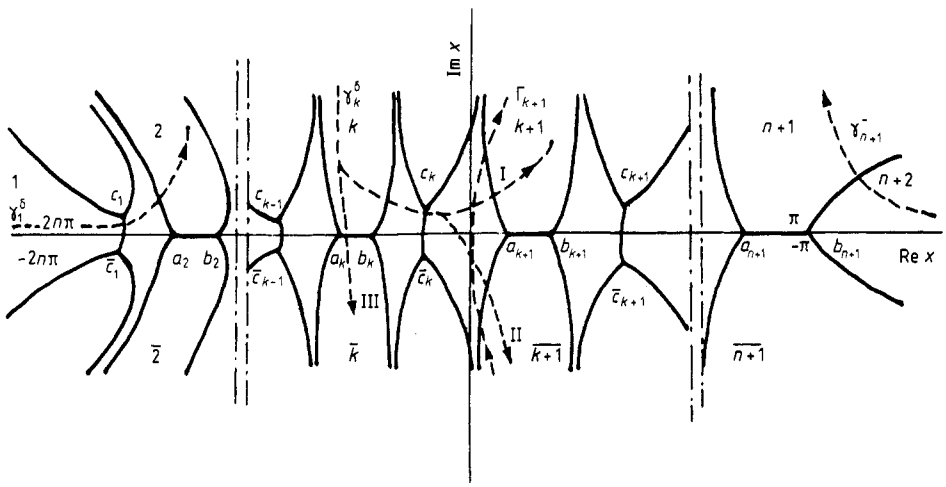

Figure 2. The Stokes graph for the potential $U_-(x)$. The possible γ paths of analytic continuations are also indicated.

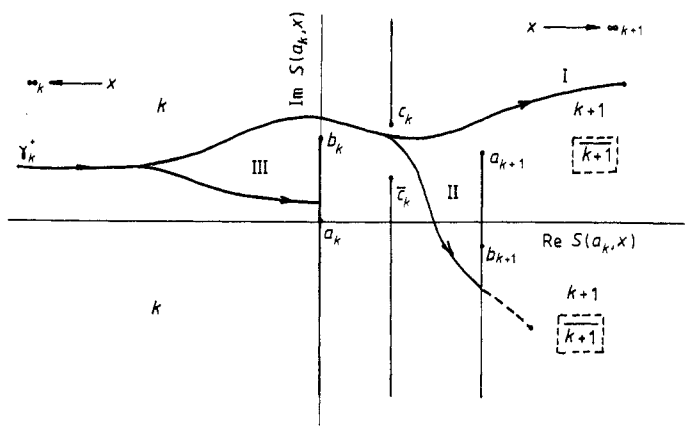

Figure 3. The γ paths in the S planes. The straight lines represent cuts.

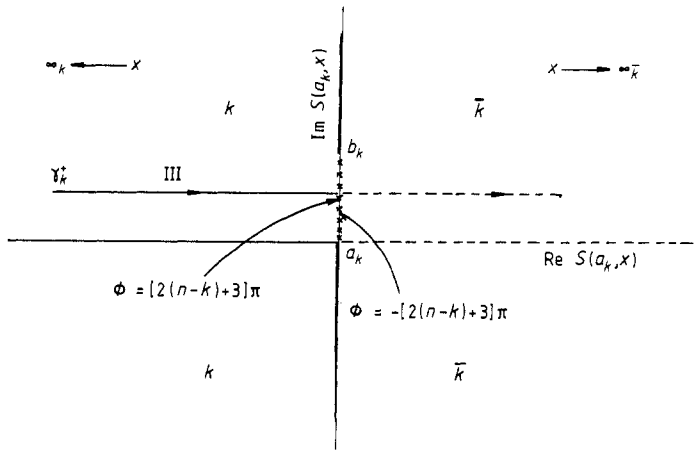

Figure 4. The γ paths in the S planes, with $\phi = \arg q(x)$.

the maxima z_k^+ of the corresponding superpotential $V(x)$ are localised in figure 2 in the vincinity of the $(c_k, \bar{c}_k)$ segments and its minima z_k^- are localised inside the intervals (a_k, b_k). From each minima z_k^-, $k = 1, \ldots, n+1$ the line Im $V(x) = 0$ emerges and goes to the infinities ∞_k or $\infty_{\bar{k}}$ of the appropriate sectors k and $\bar{k}$. On the other hand Re $V(x)$ remains negative along these lines, having its (negative) maxima at the z_k^-. Let ψ_1, ψ_{n+2} and ψ_k, $\psi_{\bar{k}}$ with $k = 2, \ldots, n+1$ be the solutions given in the form (3) and (4) in the correspondingly numbered sectors of the SG. Now it follows trivially from the properties of the solutions ψ_1 and ψ_{n+2} that the quantisation condition is

$$\psi_1(x) = C\psi_{n+2}(x). \tag{8}$$

However, the above condition cannot be used directly, since neither can ψ_1 be continued (with the help of the formulae (3)-(5)) to sector $n+2$, nor ψ_{n+2} to sector 1. The only way to get (8) explicitly is to use chains of the solutions ψ_k, $\psi_{\bar{k}}$ with $k = 2, \ldots, n+1$. That is, we can write series of the following relations

$$\begin{aligned} &\psi_1 = \alpha_1\psi_2 + \bar{\alpha}_1\psi_{\bar{2}} && \psi_k = \alpha_k\psi_{k+1} + \beta_k\psi_{\overline{k+1}} \\ &\psi_{\bar{k}} = \bar{\beta}_k\psi_{k+1} + \bar{\alpha}_k\psi_{\overline{k+1}} && k = 2, \ldots, n \\ &\psi_{n+2} = \alpha_{n+1}\psi_{n+1} + \bar{\alpha}_{n+1}\psi_{\overline{n+1}} \end{aligned} \tag{9}$$

where we have made use of the consequence of the reality of the potential $U_-(x)$, i.e. we have used the equations

$$\begin{aligned} &\psi_1(x) = \bar{\psi}_1(\bar{x}) \\ &\psi_{\bar{k}}(x) = \bar{\psi}_k(\bar{x}) \qquad k = 2, \ldots, n \\ &\psi_{n+1}(x) = \bar{\psi}_{n+1}(\bar{x}). \end{aligned} \tag{10}$$

The coefficients α_k, β_k, $k = 1, \ldots, n+1$ in (9) can be calculated as follows:

$$\begin{aligned} &\alpha_1 = \lim_{x \to \infty_2} (\psi_1(x)/\psi_2(x)) && \alpha_k = \lim_{x \to \infty_{\bar{k}+1}} (\psi_k(x)/\psi_{k+1}(x)) \\ &\beta_k = \lim_{x \to \infty_{k+1}} (\psi_k(x)/\psi_{\overline{k+1}}(x)) && k = 2, \ldots, n \\ &\alpha_{n+1} = \lim_{x \to \infty_{\overline{n+1}}} (\psi_{n+2}(x)/\psi_{n+1}(x)) \end{aligned} \tag{11}$$

where ∞_k, $\infty_{\bar{k}}$, $k=2,\ldots,n+1$, are infinity points in the appropriate sectors of the SG in figure 2. Equations (11) are consequences of the fact that all the analytic continuations needed in (11) are possible, as follows from § 2 and is shown in figures 2–4.

The relations (9) can be conveniently written in matrix form by introducing the following matrices:

$$A_1=\begin{bmatrix}\alpha_1 & \bar{\alpha}_1\\ 0 & 0\end{bmatrix} \qquad A_k=\begin{bmatrix}\alpha_k & \beta_k\\ \bar{\beta}_k & \bar{\alpha}_k\end{bmatrix} \qquad k=2,\ldots,n$$
$$A_{n+1}=\begin{bmatrix}\alpha_{n+1} & \bar{\alpha}_{n+1}\\ 0 & 0\end{bmatrix}. \tag{12}$$

We make use of (12) to transform the quantisation condition (8) into the form

$$(A_1\ldots A_n)_{11}=C(A_{n+1})_{11} \qquad (A_1\ldots A_n)_{12}=C(A_{n+1})_{12}. \tag{13}$$

The elimination of C from (13) gives the quantisation condition in the following final form:

$$(A_1A_2\ldots A_{n+1})_{12}-(A_{n+1}A_1A_2\ldots A_n)_{12}=0. \tag{14}$$

So far condition (14) is exact and can serve, at least in principle, as the equation for the energy eigenvalues. In any case it will be used to effectively calculate the GSE for our SUSY Hamiltonian.

To this end let us note that each matrix A_k, $k=1,\ldots,n+1$ is an entire function of energy and, therefore, the left-hand side of (14) has the same property. This property allows us to expand both the matrices A_k, $k=1,\ldots,n+1$, and the condition (14) into power series in E_0 around the point $E=0$. We have in this way:

$$A_k(E_0)=A_k^0+E_0A_k^1+E_0^2A_k^2+\ldots \qquad k=1,\ldots,n+1. \tag{15}$$

Since the value of the GSE E_0 is expected to be exponentially small (in comparison with, say, $\hbar|V''(z_k^{\pm})|$, where $z_k^{\pm}$ is any extremum of $V(x)$) we can truncate the series (15) at the second term and, using (14), we obtain an approximate expression for E_0

$$E_0\simeq-(A_1^0\ldots A_{n+1}^0-A_{n+1}^0A_1^0\ldots A_n^0)_{12}$$
$$\times\left(\sum_{k=1}^{n+1}(A_1^0\ldots A_k^1\ldots A_{n+1}^0-A_{n+1}^0A_1^0\ldots A_k^1\ldots A_n^0)_{12}\right)^{-1}. \tag{16}$$

At this stage the inaccuracy of (16) is easily handled and Δ, the corresponding correction to E_0 (which arises due to neglecting the third term in (15)), can be easily calculated (see appendix 3).

Now, in order to calculate the matrices A_k, we make use of (3) and (11) to obtain

$$\psi_1(x)=\mathrm{i}^n q^{-1/4}(x)\exp[(-1)^nS(a_2,x)]\psi_1(x)$$
$$\psi_k(x)=q^{-1/4}(x)\exp[(-1)^{n-k+1}S(a_k,x)]\psi_k(x) \qquad k=2,\ldots,n+1 \tag{17}$$
$$\psi_{n+2}(x)=q^{-1/4}(x)\exp[-S(b_{n+1},x)]\psi_{n+2}(x)$$

and

$$\alpha_1=\mathrm{i}^{n+1}\gamma_1 \qquad \alpha_{n+1}=-\mathrm{i}\exp[-S_-(b_{n+1},a_{n+1})]\gamma_{n+1}$$
$$\alpha_k=\mathrm{i}\exp[(-1)^{n-k+1}S_+(a_k,a_{k+1})]\gamma_k \tag{18}$$
$$\beta_k=\mathrm{i}(-1)^{n-k}\exp[(-1)^{n-k+1}S_+(a_k,a_{k+1})]\gamma_k \qquad k=2,\ldots,n$$

where

$$\gamma_k = \lim_{x \to \infty_{\overline{k+1}}} (\psi_k(x)/\psi_{k+1}(x)), \ldots \tag{19}$$

etc, and '±' subscript of $S_\pm$ denotes the integrations in the upper or lower half planes.

The expressions (17) and (18) are valid for any E and in particular for $E=0$. However, to obtain the α, β and γ we can use the 'supersymmetric' formulae (A1.1) of appendix 1 instead of (17). The reason for doing so is the explicit holomorphic dependence of the supersymmetric solutions on energy E. They are given simply as power series expansions in energy E about the point $E=0$ (see appendix 1 for details). Therefore, the matrices A_k^0, A_k^1, A_k^2, ... can be easily calculated with the help of the coefficients of the series in (A1.1). Use of the corresponding FF forms (17) would involve complicated series instead of simple integrals.

Using the supersymmetric solutions of appendix 1 we get for $E=0$:

$$\begin{aligned} &\psi_1(x) = C_1 \exp(-V(x)/\hbar) I_1^{(1)}(x) \\ &\psi_k(x) = C_k \exp(-V(x)/\hbar) I_k^{(1)}(x) \qquad k = 2, \ldots, n+1 \\ &\psi_{n+2}(x) = C_{n+2} \exp(-V(x)/\hbar) \end{aligned} \tag{20}$$

and

$$\begin{aligned} &\alpha_1^0 = (C_1/C_2)(I_0^{(1)}(\infty_{\overline{2}})/I_2^{(1)}(\infty_{\overline{2}})) \\ &\alpha_k^0 = (C_k/C_{k+1})(I_k^{(1)}(\infty_{\overline{k+1}})/I_{k+1}^{(1)}(\infty_{\overline{k+1}})) \\ &\beta_k^0 = (C_k/\bar{C}_{k+1})(I_k^{(1)}(\infty_{k+1})/I_{k+1}^{(1)}(\infty_{k+1})) \qquad k = 2, \ldots, n \\ &\alpha_{n+1}^0 = (C_{n+2}/C_{n+1})(1/I_{n+1}^{(1)}(\infty_{\overline{n+1}})) \end{aligned} \tag{21}$$

where the quantities $I_k^{(1)}$, $k=1, \ldots, n+1$ have been defined in appendix 1. Of course, the coefficients (21) define the matrices A_k^0 in (15). On the other hand, the A_k^1 in (15) can be calculated using (18) and the solutions (A1.1) of appendix 1. In this way we obtain (see (A1.6) and (A1.7) for details):

$$(A_k^1)_{11} = \alpha_k^0 \left[\frac{(-1)^{n-k}}{\hbar} \left(\int_{a_k}^{\infty_k} + \int_{a_{k+1}}^{\infty_{k+1}} \right) q^{-1/2} - \frac{2}{\hbar^2} \left(\frac{I_k^{(3)}(\infty_{\overline{k+1}})}{I_k^{(1)}(\infty_{\overline{k+1}})} - \frac{I_{k+1}^{(3)}(\infty_{\overline{k+1}})}{I_{k+1}^{(1)}(\infty_{\overline{k+1}})} \right) \right]$$

$$(A_k^1)_{12} = \beta_k^0 \left[\frac{(-1)^{n-k}}{\hbar} \left(\int_{a_k}^{\infty_k} + \int_{a_{k+1}}^{\infty_{\overline{k+1}}} \right) q^{-1/2} - \frac{2}{\hbar^2} \left(\frac{I_k^{(3)}(\infty_{k+1})}{I_k^{(1)}(\infty_{k+1})} - \frac{I_{k+1}^{(3)}(\infty_{k+1})}{I_{k+1}^{(1)}(\infty_{k+1})} \right) \right]$$

where $k = 2, \ldots, ,$ and (22)

$$(A_1^1)_{11} = \alpha_1^0 \left[\frac{(-1)^{n-1}}{\hbar} \left(\int_{a_2}^{\infty_1} + \int_{a_2}^{\infty_2} \right) q^{-1/2} - \frac{2}{\hbar^2} \left(\frac{I_1^{(3)}(\infty_{\overline{2}})}{I_1^{(1)}(\infty_{\overline{2}})} - \frac{I_2^{(3)}(\infty_{\overline{2}})}{I_2^{(1)}(\infty_{\overline{2}})} \right) \right]$$

$$(A_{n+1}^1)_{11} = \alpha_{n+1}^0 \left[\frac{1}{\hbar} \left(\int_{b_{n+1}}^{\infty_{n+2}} + \int_{a_{n+1}}^{\infty_{n+1}} \right) q^{-1/2} - \frac{2}{\hbar^2} \left(I_{n+2}^{(2)}(\infty_{\overline{n+1}}) - \frac{I_{n+1}^{(3)}(\infty_{\overline{n+1}})}{I_{n+1}^{(1)}(\infty_{\overline{n+1}})} \right) \right]$$

where all the quantities on the right-hand side of (22) are defined in appendix 1. Let us now observe that

$$A_k^r = C_{(k)} B_k^r C_{(k+1)}^{-1} \qquad r = 0, 1 \qquad k = 1, \ldots, n+1 \tag{23}$$

where

$$C_{(k)} = \begin{bmatrix} C_k & 0 \\ 0 & C_k \end{bmatrix} \qquad k = 1, \ldots, k+1$$

and the B_k^r introduced above are defined only by the integrals present in (21) and (22). Substitution of (21)-(23) into (16) gives

$$E_0 = -(B_1^0 \dots B_{n+1}^0 - B_{n+1}^0 B_1^0 \dots B_n^0)_{12} \times \left(\sum_{k=1}^{n+1} (B_1^0 \dots B_k^1 \dots B_{n+1}^0 - B_{n+1}^0 B_1^0 \dots B_k^1 \dots B_n^0)_{12} \right)^{-1}. \tag{24}$$

The algebraic operations involved in the above formula can be performed effectively, leaving us with the rather simple result (see appendix 2 for details)

$$E_0^{-1} = -\sum_{k=1}^{n} D_k + \sum_{k=2}^{n} 2\mathrm{i}\ \mathrm{Im}\ D_k \sum_{r=1}^{k-1} (I_{k+1}/I_k)(B_r^0)_{12} - D_{n+1}/(B_{n+1}^0)_{12} + (2\ \mathrm{Re}\ D_{n+1}/(B_{n+1}^0)_{12}) \sum_{r=1}^{n} (I_{r+1}/I_{n+1})(B_r^0)_{12} \tag{25}$$

where the I_k and D_k, $k = 1, \dots, n+1$, are defined in appendix 2.

We would like to stress at the moment that so far the above formula takes into account only the assumption that E_0 is small in comparison with any of the quantities $\hbar |V''(z_k^{\pm})|$. However, in the form (25) the formula is still rather complicated. One can expect some simplifications in (25) if the exact integrals were replaced by their approximations. The simple approximations seem to be the asymptotic forms for $\hbar \to 0$. Taking the limit we get the following estimates for the leading quantities in (25):

$$I_k = \int_{\infty_{\bar{k}}}^{\infty_k} \exp(2V/\hbar)\, \mathrm{d}y \sim \mathrm{i}(\pi\hbar/(V_k^-)'')^{1/2} \exp(2V_k^-/\hbar)$$
$$I_k^{(1)}(\infty_{k+1}) \equiv I_k^{(1)} \sim (\pi\hbar/|(V_k^+)''|)^{1/2} \exp(2V_k^+/\hbar) \tag{26}$$

and

$$D_k \sim Q_{k+1} I_k^{(1)}/I_{k+1} - (\pi/\hbar)|(V_{k+1}^-)''(V_k^+)''|^{-1/2} \exp[2(V_k^+ - V_{k+1}^-)/\hbar] \qquad k = 1, \dots, n$$
$$(2\mathrm{i}\ \mathrm{Im}\ D_k/I_k) \sim (Q_k/I_k) - (Q_{k+1}/I_{k+1}) + (1/\hbar^2)[(\pi\hbar/(V_k^-)'')^{1/2} \exp(-2V_k^-/\hbar) + (\pi\hbar/(V_{k+1}^-)'')^{1/2} \exp(-2V_{k+1}^-/\hbar)] \qquad k = 2, \dots n \tag{27}$$
$$2\ \mathrm{Re}\ D_{n+1} \sim (Q_{n+1}/I_{n+1}) + (1/\hbar^2)(\pi\hbar/(V_{n+1}^-)'')^{1/2} \exp(-2V_{n+1}^-/\hbar)$$

and dropping all the terms which are of higher order (or exponentially small) in $\hbar$. In the above formulae $(V_k^{\pm})'' \equiv V''(z_k^{\pm})$ and

$$Q_k = \frac{(-1)^{n-k+1}}{\hbar} \int_{\Gamma_{k+1}} q^{-1/2}\, \mathrm{d}y \tag{28}$$

with the integration path Γ_{k+1} shown in figure 2.

The estimations (26) and (27) have been done whith the help of the saddle-point method by integrations along the lines $\mathrm{Im}\ V(x) = 0$, described previously.. They are valid if $\hbar \ll |V''(z_k^-)|(z_k^+ - z_k^-)^2 \simeq V(z_k^+) - V(z_k^-)$ for each extremum $z_k^{\pm}$ of $V(x)$ (see appendix 3 for details).

Noticing now that all the terms proportional to the Q_k mutually cancel, we get at last

$$E_0 \simeq \frac{\hbar}{2\pi} \left(\sum_{k=1}^{n} \sum_{r=0}^{n-k} |(V''(z_k^+)\, V''(z_{k+r+1}^-)|^{-1/2} \exp[2(V_k^+ - V_{k+r+1}^-)/\hbar] \right)^{-1} \tag{29}$$

or

$$E_0 \simeq (\hbar/2\pi)|V''(z_{l_0}^+)\, V''(z_{k_0}^-)|^{1/2} \exp[-2/\hbar)(V(z_{l_0}^+) - V(z_{k_0}^-))]. \tag{30}$$

In the above formula $z_{l_0}^+$ and $z_{k_0}^-$ are such extrema of $V(x)$ for which the difference $V(z_l^+) - V(z_k^-)$ is maximal under the condition that $k \geq l$, i.e. the minimum is *always* to the right of the maximum.

The above result needs some comment. It follows that E_0 is determined by the relative maximum of the differences $V(z_l^+) - V(z_k^-)$ defined above rather than by the absolute maximum. This can be easily understood as follows. Let us make some minimum of the superpotential $V(x)$, say z_k^-, arbitrarily large (in absolute value). It causes the nearest potential barriers a, b, placed on both the sides of the minimum z_k^- (see figure 1) to also tend to infinity. These two barriers divide our broken superpotential $V(x)$ into three parts. The left and the middle parts, each having an odd number of extrema ($2k-3$ for the left and 1 for the middle) become supersymmetric in the limit of the infinitely growing barriers. On the other hand, in the part to the right of both the barriers the supersymmetry remains broken, since it contains an even number of the extrema. It follows then, that in the left and in the middle part of $V(x)$, the energy E_0 (given by (30), where we have to put $z_{k_0}^- = z_k^-$) should vanish in the limit considered, while in the right part it should remain finite. It is clear that it is possible only when, in (30), z_l^+ is placed to the left of z_k^-, rather than to the right.

The following remarks are in order: (i) E_0 as given by (30) vanishes exponentially for $\hbar \to 0_+$ and therefore the asymptotic approximations (26) and (27) are in accordance with the assumption that E_0 is small; (ii) equation (30) can be applied not only to obtain the numerical value of E_0 (under the conditions of appendix 3) but it also gives the asymptotic expression for E_0 in the limit $\hbar \to 0$; (ii) equation (25) has been obtained using *the exact* solutions to the SE (i.e. not the asymptotic ones for $\hbar \to 0$)—the limit $\hbar \to 0$ has been taken only in the final expression (25) without any contradiction with the previous assumptions (see remark (i) above).

4. Final remarks

We conclude with some remarks. Firstly, let us note that although our result for E_0 has the same form as found previously (Salomonson and Van Molten 1982, Giler *et al* 1985), it now has the precise meaning given above (see (30) and the comment immediately following). It improves our earlier result obtained by another method (Giler *et al* 1985). Secondly, the result (30) would be exactly the same (as it should be), if we used the solutions (A1.1), taken at $E = 0$, from the very beginning. However, we preferred the FF forms instead of the 'pure' supersymmetric ones, because of the apparently asymptotic character of the former for $\hbar \to 0$. On the other hand, the use of the simpler asymptotic formulae for the FF solutions was impossible in our case, since, in the limit $\hbar \to 0$, their necessary analytic continuations to different sections of the SG became impossible (it was because the segments (a_k, b_k) and $(c_k, \bar{c}_k)$, through which these analytic continuations were going, contracted to points when $\hbar \to 0$ (see figure 2)).

Further, it is not difficult to see that the above method can be generalised to superpotentials which also have complex zeros. Equation (30) gives us the exponentially small expression for the GSE. It follows, then, that the perturbative expansion for the GSE, being asymptotic, vanishes. This is a characteristic property of supersymmetry. Let us note, however, that according to the general theorems it is possible only if SUSY is unbroken at the tree level. Again the same conclusion follows from the formalism developed above.

Let us note also that our basic approximate formula (15) for the GSE clearly distinguishes between the broken and unbroken cases of the superpotential $V(x)$. For the unbroken superpotential it gives *exactly* zero for the GSE as it should do, since zero is the exact value of the GSE in this case. On the other hand the asymptotic series for the GSE vanish identically in both cases. It proves, therefore, that the result (30) for the broken symmetry case cannot be regarded as merely a matter of chance.

Finally, let us note for completeness that for the higher-lying eigenstates of, say, a double-well symmetric SUSY potential, one can deduce the usual formula for the energy splitting.

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Appendix 1

For the SE with superpotential there exists for every sector of the SG (figure 2) a supersymmetric analogue of the FF solution (3). Namely, we have (Giler *et al* 1985)

$$\psi_k(x) = C_k \exp(-V(x)/\hbar) \sum_{r\geqslant 0} (-2E/\hbar^2)^r I_k^{(2r+1)}(x) \qquad k = 1, \ldots, n+1$$
$$\psi_{n+2}(x) = C_{n+2} \exp(-V(x)/\hbar) \sum_{r\geqslant 0} (-2E/\hbar^2)^r I_{n+2}^{(2r)}(x) \tag{A1.1}$$

where C_k, $k = 1, \ldots, n+2$ are some normalisation constants and

$$I_k^{(2r+1)}(x) = \int_{\tilde{\gamma}_k(x)} \mathrm{d}y_1 \ldots \int_{\tilde{\gamma}_k(y_{2r})} \mathrm{d}y_{2r+1} \exp[2(V_1 - V_2 + \ldots - V_{2r} + V_{2r+1})/\hbar]$$
$$k = 1, \ldots, n+1 \qquad r = 0, 1, 2, \ldots$$
$$I_{n+2}^{(2r)}(x) = \int_{\tilde{\gamma}_{n+2}(x)} \mathrm{d}y_1 \ldots \int_{\tilde{\gamma}_{n+2}(y_{2r-1})} \mathrm{d}y_{2r} \exp[2(V_1 - V_2 + \ldots + V_{2r-1} - V_{2r})/\hbar]$$
$$r = 0, 1, 2, \ldots \tag{A1.2}$$

with $V_r = V(y_r)$ and $I_{n+2}^{(0)}(x) \equiv 1$.

The paths $\tilde{\gamma}$ in (A1.2) are analogous to the paths γ in (4). The $\tilde{\gamma}_k$ path starts from the k infinity and continues to the point x in such a way as to maintain $\mathrm{Re}(V(y_{2i}) - V(y_{2i+1})) \geqslant 0$ along the whole path. It should be noted that $\mathrm{Re}\, V(x) \to -\infty$ when $x \to \infty_k$ in every sector k (or $\bar{k}$) of the SG except the sector $n+2$ where $\mathrm{Re}\, V(x) \to +\infty$.

Both the forms (A1.1) and (3) represent the same solutions to the SE in the corresponding sectors of the Stokes graph in figure 2. Let us note also that each solution to the SE can be given holomorphic dependence on the energy E by multiplying it by suitable constant $c(E)$. The main difference between the forms (A1.1) and (3) is that this holomorphic dependence is explicitly given in the supersymmetric form (A1.1) in contrast to the FF form (3) where such a holomorphic dependence is completely obscure. Using the forms (A1.1) one can easily show that $\psi_k(x)$, $k = 1, \ldots, n+2$, can be chosen as entire functions of E. Namely, the coefficients $I_r^{(2r+1)}(x)$,

$k=1,\ldots,n+1$, and $I_{n+2}^{(2r)}(x)$, $r=1,2,\ldots$, in (A1.2) can be bounded as follows:

$$|I_k^{(2r+1)}(x)| \leqslant \frac{1}{r!}\int_{\tilde{\gamma}_k(x)} |dy| \exp[2\mathrm{Re}(V(y)/\hbar)]\left(\int_{\tilde{\gamma}_k(x)} |dy| F_k(y)\right)^r \qquad k=1,\ldots,n+1$$
$$|I_{n+2}^{(2r)}(x)| \leqslant \frac{1}{r!}\left(\int_{\tilde{\gamma}_{n+2}(x)} |dy| F_{n+2}(y)\right)^r \qquad r=0,1,2,\ldots \tag{A1.3}$$

where

$$F_k(x) = \exp[-2\,\mathrm{Re}(V(x)/\hbar)]\int_{\tilde{\gamma}_k(x)} \exp[2\mathrm{Re}(V(y)/\hbar)]|dy| \qquad k=1,\ldots,n+1$$
$$F_{n+2}(x) = \exp[-2\,\mathrm{Re}(V(x)/\hbar)]\int_{\tilde{\gamma}_{n+2}(x)} \exp[2\,\mathrm{Re}(V(y)/\hbar)]|dy|. \tag{A1.4}$$

Let us note that the $F_k(x)$ are finite when $x\to\infty_k$, $k=1,\ldots,n+2$, and in fact, vanish like $1/V'(x)$ in this limit.

The estimation (A1.3) also yields the following bounds for the series in (A1.1):

$$\left|\sum_{r\geqslant 0}(-2E/\hbar^2)^r I_k^{(2r+1)}(x)\right| \leqslant \int_{\tilde{\gamma}_k(x)} |dy| \exp[2\,\mathrm{Re}(V(y)/\hbar)] \exp\left(|2E/\hbar^2| \times \int_{\tilde{\gamma}_k(x)} |dy| F_k(y)\right) \qquad k=1,\ldots,n+1$$
$$\left|\sum_{r\geqslant 0}(-2E/\hbar^2)^r I_{n+2}^{(2r)}(x)\right| \leqslant \exp\left(|2E/\hbar^2|\int_{\tilde{\gamma}_{n+2}} |dy| F_k(y)\right) \tag{A1.5}$$

which prove the holomorphicity of each series in (A1.1) in the whole complex E plane for $|\hbar|>0$. However, one should have in mind that by changing E in the E plane one also changes the Stokes graph and the corresponding solutions (A1.1) should follow these changes.

Obviously, the solutions (A1.1) are not in general appropriate for calculations in the limit $\hbar\to 0$. But they appear to be extremely useful in the specific case $E=0$, because of the obvious abbreviations of the series in (A1.1). These enable us to obtain in this case the explicit expressions (21) and (22) for the matrices A_k^0 and A_k^1 respectively. Since the method of obtaining the expressions (21) is obvious we sketch only the corresponding way for the expressions (22). To get, for example, $(A_k')_{11}$ we proceed as follows. From (8) we have

$$\frac{d\alpha_k}{dE} = \alpha_k \lim_{x\to\infty_{\overline{k+1}}} \frac{d\ln(\psi_k(x)/\psi_{k+1}(x))}{dE}. \tag{A1.6}$$

Using (A1.1) for $E=0$ we arrive at

$$\frac{d\alpha_k^0}{dE} = \alpha_k^0\left[\frac{d\ln(C_k/C_{k+1})}{dE}\bigg|_{E=0} + \frac{2}{\hbar^2}\left(\frac{I_{k+1}^{(3)}(\infty_{\overline{k+1}})}{I_{k+1}^{(1)}(\infty_{\overline{k+1}})} - \frac{I_k^{(3)}(\infty_{\overline{k+1}})}{I_k^{(1)}(\infty_{\overline{k+1}})}\right)\right]. \tag{A1.7}$$

On the other hand the first term in the square bracket in (A1.7) can be calculated with

the help of the solutions (17), to get

$$\left.\frac{\mathrm{d}\ln(C_k/C_{k+1})}{\mathrm{d}E}\right|_{E=0}=\frac{(-1)^{n-k}}{\hbar}\left(\int_{a_k}^{\infty_k}+\int_{a_{k+1}}^{\infty_{k+1}}\right)q^{-1/2}. \tag{A1.8}$$

Appendix 2

To obtain the expression (25) for E_0 the following properties of the matrices B_k^0 and B_k^1 should be taken into account:

$$\begin{aligned}&(B_k^0)_{j1}+(B_k^0)_{j2}=1 \qquad (B_k^0)_{2j}=(\overline{B_k^0})_{1j+1} \qquad j=1,2\\&(B_k^0)_{22}-(B_k^0)_{12}=I_k^{(1)}(\infty_{\bar{k}})/I_{k+1}^{(1)}(\infty_{\overline{k+1}})\equiv I_k/I_{k+1} \qquad k=1,\ldots,n\\&(B_{n+1}^0)_{11}+(B_{n+1}^0)_{12}=0.\end{aligned} \tag{A2.1}$$

Then for the numerator in the right-hand side of (24) we have

$$\begin{aligned}&(B_1^0\ldots B_{n+1}^0-B_{n+1}^0B_1^0\ldots B_n^0)_{12}\\&\quad=(B_1^0\ldots B_n^0)_{11}(B_{n+1}^0)_{12}-(B_{n+1}^0)_{11}(B_1^0\ldots B_n^0)_{12}\\&\quad=[(B_1^0\ldots B_n^0)_{11}+(B_1^0\ldots B_n^0)_{12}](B_{n+1}^0)_{12}\\&\quad=(B_1^0\ldots B_{n-1}^0)_{1k}[(B_n^0)_{k1}+(B_n^0)_{k2}](B_{n+1}^0)_{12}\\&\quad=[(B_1^0\ldots B_{n-1}^0)_{11}+(B_1^0\ldots B_{n-1}^0)_{12}](B_{n+1}^0)_{12}=(B_{n+1}^0)_{12}\equiv b.\end{aligned} \tag{A2.2}$$

Similarly, for the kth term of the denominator in (24) we get

$$\begin{aligned}&(B_1^0\ldots B_k^1\ldots B_{n+1}^0)_{12}-(B_{n+1}^0B_1^0\ldots B_k^1\ldots B_n^0)_{12}\\&\quad=[(B_1^0\ldots B_k^1)_{11}+(B_1^0\ldots B_k^1)_{12}]b\\&\quad=[(B_1^0\ldots B_{k-1}^0)_{11}D_k+(B_1^0\ldots B_{k-1}^0)D_k]b\\&\quad=\{\mathrm{Re}\,D_k+[(B_1^0\ldots B_{k-1}^0)_{11}-(B_1^0\ldots B_{k-1}^0)_{12}]\mathrm{i}\,\mathrm{Im}\,D_k\}b\\&\quad=\{\mathrm{Re}\,D_k+[(B_1^0\ldots B_{k-2}^0)_{12}+(B_1^0\ldots B_{k-2}^0)_{11}][(B_{k-1}^0)_{11}-(B_{k-1}^0)_{12}]\\&\qquad+(B_1^0\ldots B_{k-2}^0)_{12}[(B_{k-1}^0)_{21}-(B_{k-1}^0)_{22}]\mathrm{i}\,\mathrm{Im}\,D_k\}b\\&\quad=\{D_k-2\mathrm{i}\,\mathrm{Im}\,D_k[(B_{k-1}^0)_{12}+(B_1^0\ldots B_{k-2}^0)_{12}(I_{k-1}/I_k)]\}b\\&\quad=\{D_k-2\mathrm{i}\,\mathrm{Im}\,D_k\sum_{r=1}^{n}(B_r^0)_{12}(I_{r+1}/I_k)\}b\end{aligned} \tag{A2.3}$$

where $D_k=(B_k^1)_{11}+(B_k^1)_{12}$, $k=1,\ldots,n$. For the last term in the denominator in (24) we get instead

$$\begin{aligned}&(B_1^0\ldots B_n^0B_{n+1}^1-B_{n+1}^1B_1^0\ldots B_n^0)_{12}\\&\quad=(B_1^0\ldots B_n^0)_{11}(B_{n+1}^1)_{12}-(B_{n+1}^1)_{11}(B_1^0\ldots B_n^0)_{12}\\&\quad=(D_{n+1}-2\,\mathrm{Re}\,D_{n+1}\sum_{r=1}^{n}(B_r^0)_{12}(I_r/I_{r+1})\end{aligned} \tag{A2.4}$$

where $D_{n+1}=(B^1_{n+1})_{12}$. The expression (25) for E_0 now follows directly from (A2.2)-(A2.4).

Appendix 3

The correction Δ to E_0 can be estimated by noticing that it arises due to neglecting the third term in (15). Keeping this term in (15) and putting there $E=E_0+\Delta$, where E_0 is given by (16) we get for Δ the result

$$\Delta/E_0=-E_0\sum_{k=1}^{n+1}[A^0_1\dots A^2_k\dots A^0_nA^0_{n+1}-A^0_{n+1}A^0_1\dots A^2_k\dots A^0_n]_{12}$$
$$\times\Bigg(\sum_{k=1}^{n+1}[A^0_1\dots A^1_k\dots A^0_nA^0_{n+1}-A^0_{n+1}A^0_1\dots A^1_k\dots A^0_n]_{12}$$
$$+4E_0\sum_{\substack{k,l=1\\k<l}}^{n+1}[A^0_1\dots A^1_k\dots A^1_l\dots A^0_nA^0_{n+1}$$
$$-A^0_{n+1}A^0_1\dots A^1_k\dots A^1_l\dots A^0_n]_{12}\Bigg)^{-1}. \tag{A3.1}$$

However, we are not going to perform detailed calculations in (A3.1) to explicitly check its validity (i.e. that $|\Delta/E_0|\ll 1$). We are convinced about the smallness of (A3.1) since the assumption that $|E_0/[\hbar V''(z^{\pm}_k)]|$ is small (for any $z^{\pm}_k$) is confirmed by (30) where it is determined by the quantity

$$\exp[-(2/\hbar)(V(z^+_{l_0})-V(z^-_{k_0}))]. \tag{A3.2}$$

For the quantity (A3.2) to be small requires the following relation to be satisfied:

$$\hbar\ll 2(V(z^+_{k-1})-V(z^-_k)). \tag{A3.3}$$

However, (A3.3) is exactly the condition which controls the validity of the approximations (26) and (27). Namely, if (26) and (27) are to be good the second terms in their asymptotic expansions in $\hbar$ for $\hbar\to 0$ have to be much smaller than the first terms. Thus, for example, for the integrals I_k in (26) this condition gives

$$\hbar\ll 16|V''(z^-_k)|^2/|V^{(4)}(z^-_k)+\tfrac{7}{3}[V^{(3)}(z^-_k)]^2/V''(z^-_k)|$$
$$\simeq|V''(z^-_k)|(z^+_{k-1}-z^-_k)^2\simeq V(z^+_{k-1})-V(z^-_k) \tag{A3.4}$$

where, in order to make the two last steps, we have made use of the obvious approximations

$$V^{(3)}(z^-_k)\simeq[V''(z^-_k)-V''(z^+_{k-1})]/[z^-_k-z^+_{k-1}]$$
$$\simeq 2V''(z^-_k)/[z^-_k-z^+_{k-1}]$$

and

$$V^{(4)}(z^-_k)\simeq[V^{(3)}(z^-_k)-V^{(3)}(z^+_{k-1})]/[z^-_k-z^+_{k-1}]$$
$$\simeq 2V^{(3)}(z^-_k)/[z^-_k-z^+_{k-1}]\simeq 4V''(z^-_k)/[z^-_k-z^+_{k-1}]^2. \tag{A3.5}$$

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